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Bounds on the Root-Mean-Square Miss of Radar-Guided Missiles Against Sinusoidal Target Maneuvers

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In preliminary analysis of guided systems, it is required to assess the miss distance performance from some small set of parameters. Previous papers presented analytical formulas of bounds on the root-mean-square miss by a radar-guided missile against step- and exponential-type maneuvers. This paper presents formulas against harmonic-sinusoidal maneuvers. Moreover, a type of maneuver that is not treated in the literature, the continuously sinusoidal maneuver with uncertain-random frequency and magnitude, is introduced, and bound on the miss is presented. The formulas use a set of core parameters that affect the miss distance; thus, they can be used for synthesis and analysis of the performance of radar-guided tactical missiles. The bound is derived subject to assumption that the missile guidance law and estimator are fully matched to the missile dynamics, the target maneuver, and the glint noise. The glint is the dominant noise source, the missile applies frequency agility, there is no blind range or missile acceleration limit, and the terminal phase period is sufficiently long so that initial conditions fade away. No system can achieve smaller root-mean-square miss distance than the one presented, subject to the stated assumptions.

I. Introduction

THERE are many publications that give approaches and formulas to estimate the root-mean-square (rms) miss distance [1–4]. In a previous publication [5], an approach to derive estimated and analytical closed-form formulas of bounds on the rms miss of a radar-guided missile against step- and exponential-type maneuvers has been presented. These formulas give assessment of the bound on the rms miss [the greatest lower bound (GLB)] of a radar-guided missile by some minimal set of parameters (a set of core parameters).

By a set of core parameters, two types of parameters are addressed:

1) The first set of parameters is not in the hands of the system engineer: the size, acceleration level, and velocity of the target; the period of the encounter, the period of the sine-wave maneuver, and the respective distributions.

2) The second set of parameters is in the hands of the system engineer: the sampling rate and the algorithms implementation (frequency agility implementation in the radar, and the guidance laws).

In [5], bounds on the rms had been presented for step- and exponentially correlated target maneuvers. This paper is an extension of [5] for harmonic target maneuvers. Guidance laws against a sinusoidal maneuver are dealt with in [6,7]. Here, the same assumptions, approach, nomenclature, notation, and method are used to derive the bounds, as in [5].

Parts of the theoretical derivations are duplicated here and updated for the clarity of presentation.

Specifically, we derive GLB of the rms miss for 1) the abruptly starting sine-wave maneuver, 2) the continuous harmonic wave maneuver, and 3) the nearly constant turn-rate (CT) maneuver.

The following is a summary of the assumptions used in derivation of the bounds:

1) There is a linear scenario.
2) The dominant noise is the glint.
3) The radar seeker applies frequency agility.
4) The guidance law is matched to missile and the target dynamics.
5) The missile has no acceleration constraint.
6) There is no blind range.
7) The terminal phase period is sufficiently long so that initial conditions (heading error, etc.) fade away.

The importance of deriving bounds in general, and especially GLBs, is 1) to have the ultimate achievable performance (nobody can achieve better performance than the bound), 2) to have a measure of an achieved performance from the optimal/best possible performance, and 3) if the bound has analytic expression, the system engineer can tradeoff the different variables/parameters to optimize performance in pursuit of a cost-effective design, and more.

No system can achieve smaller rms miss distance than the presented GLBs, subject to the stated assumptions. Any deviation from these assumptions will result in an increase of the miss distance above the GLB.

II. Problem Statement and Approach to Solution

For consistency and clarity of presentation, most of this section is repeated from [5]. The most natural domain to pose the presented problem would be the domain of stochastic zero-sum differential games (noncooperative dynamic games) [8–12].

So formally, the solution/value of (heuristically)

$$\min_{\text{missile's parameters}} \max_{\text{target's parameters}} J(\text{miss}) \tag{1}$$

is sought, where $J()$ is the objective; i.e., the miss distance and energy expenditure.

The deterministic case of zero-sum differential games is treated in [12], where bounds on the deterministic miss called guaranteed miss are presented. It would be most appropriate to use the solution of the stochastic two-person zero-sum differential game with noisy measurements and system driving noise (winds gusts, pilot behavior, etc.). This case is dealt with in [13]; however, the solution is infinite-dimensional.

However, today's targets do not perform evasive maneuvers based on the differential game theory; thus, although it is interesting, it does not have immediate practical value.

Following the preceding discussion, in this work, it is assumed that the target performs a specific maneuver that is known to the missile. The missile applies the optimal guidance law and estimator based on one-sided optimization. The missile has no acceleration limit; that is, the linear optimal control problem is solved. This way, the GLB on the rms miss is derivable. Any deviation from the stated assumption leads to an increase of the rms miss above the bound.

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The bound depends on a small set of parameters: the core parameters. It is our belief that the presented core parameter set is the minimal set of parameters that gives the bound on the miss. Neither the minimality nor the uniqueness of this set is proved in the paper.

The glint is the dominant noise contributor to the miss from all the noises and disturbances (i.e., the effects of the thermal noise, inertial sensors noises, etc. [1] are not considered). This is further justified by the fact that the noises, other than glint (range dependent and independent angular noises), have zero contribution to the miss [3,14].

It is assumed that the missile implements frequency agility to decorrelate the glint noise, and the missile designer manages the sampling rate. In this case, the deterministic miss is negligible and, for all practical purposes, zero; thus, the rms miss is only due to measurement noise: the glint.

III. Derivation of Greatest Lower Bound

A. Linear Quadratic Gaussian Optimal Control Problem

A missile that implements guidance, control, and estimation based on the linear quadratic Gaussian optimal control (LQG) problem [11,15] is considered. The problem considered is a special case of this issue. The following is the presentation, for the completeness of the presentation in this paper, of the LQG problem and its solution.

We consider the linear stochastic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \Gamma w(t), \qquad x(t_o) = x_o \\ z(t) &= Cx(t) + v(t) \end{aligned} \tag{2}$$

where $x(t)$ is the state vector, $z(t)$ is the measurement, $u(t)$ is the input, $w(t)$ and $v(t)$ are the white Gaussian stochastic processes representing the system driving noise and the measurement noise, respectively, $x(t_o)$ is a Gaussian random vector, and

$$\begin{gathered} E[x_o] = \bar{x}_o, \qquad E[w(t)] = 0, \qquad E[v(t)] = 0 \\ E[w(t)w(\tau)^T] = W\delta(t-\tau), \qquad E[v(t)v(\tau)^T] = V\delta(t-\tau) \\ E[w(t)v(\tau)^T] = 0, \qquad E[w(t)x_o^T] = 0, \qquad E[v(t)x_o^T] = 0 \\ E\{[x_o - E(x_o)][x_o - E(x_o)^T]\} = Q_o \end{gathered} \tag{3}$$

All vectors and matrices are of appropriate dimensions.

The problem being considered here is finding the optimal control $u^*(t)$ as a functional of $\{z(t), t_o \le t \le t_f\}$ that minimizes the quadratic criterion:

$$J = \frac{1}{2}E\left[x^T(t_f)Gx(t_f) + \int_{t_o}^{t_f}[z^t(t)Q_c z(t) + u^T(t)Ru(t)]\,\mathrm{d}t\right] \tag{4}$$

The solution is the cascade of a Kalman filter and deterministic optimal controller [11,15]. This solution is based on the certainty equivalence principle and the separation theorem. More elaborated derivation, rationale. and motivation of the results in this section are presented in Appendix A.

The Kalman filter is

$$\begin{aligned} &\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(t)[z(t) - C\hat{x}(t)], \qquad \hat{x}(t_o) = \bar{x}_o \\ &K(t) = Q(t)C^TV^{-1} \\ &\dot{Q}(t) = AQ(t) + Q(t)A^T + \Gamma W\Gamma^T - Q(t)C^TV^{-1}CQ(t), \\ &\quad Q(t_o) = Q_o \end{aligned} \tag{5}$$

and the optimal controller (guidance law) is

$$\begin{aligned} &u^*(t) = -F(t)\hat{x}(t) \qquad F(t) = R^{-1}B^TP(t) \\ &-\dot{P} = P(t)A + A^TP(t) + C^TQ_cC - P(t)BR^{-1}B^TP(t), \\ &\quad P(t_f) = G \end{aligned} \tag{6}$$

To derive the bound, the estimation error is considered [15]:

$$e(t) = x(t) - \hat{x}(t) \tag{7}$$

and the dynamic equation of the combined system is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix} &= \begin{bmatrix} A - K(t)C & 0 \\ K(t)C & A - BF(t) \end{bmatrix}\begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} \Gamma & -K(t) \\ 0 & K(t) \end{bmatrix}\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \\ \begin{bmatrix} e(t_o) \\ \hat{x}(t_o) \end{bmatrix} &= \begin{bmatrix} x(t_o) - \hat{x}(t_o) \\ \bar{x}_o \end{bmatrix} = \begin{bmatrix} x_o \\ 0 \end{bmatrix} \end{aligned} \tag{8}$$

We are interested in

$$\begin{aligned} E[x^T(t)Gx(t)] &= \mathrm{trace}\{G[\bar{x}(t)\bar{x}^T(t) + Q_{\hat{x}\hat{x}}(t) + Q(t)]\} \\ &\ge \mathrm{trace}\{G[Q_{\hat{x}\hat{x}}(t) + Q(t)]\} \ge \mathrm{trace}[GQ(t)] \end{aligned} \tag{9}$$

where

$$\begin{aligned} Q(t) &= E(\{e(t) - E[e(t)]\}\{e(t) - E[e(t)]\}^T) \\ Q_{\hat{x}\hat{x}} &= E(\{\hat{x}(t) - E[\hat{x}(t)]\}\{\hat{x}(t) - E[\hat{x}(t)]\}^T) \\ \bar{x}(t) &= E[x(t)] \end{aligned} \tag{10}$$

If it is assumed that $E[x_o] = \bar{x}_o = 0$, then $\bar{x}(t) = E[x(t)] = 0$. Furthermore, from [15],

$$\begin{aligned} &\dot{Q}_{\hat{x}\hat{x}}(t) = [A - BF(t)]Q_{\hat{x}\hat{x}}(t) + Q_{\hat{x}\hat{x}}(t)[A - BF(t)]^T \\ &\quad + K(t)VK^T(t); \\ &Q_{\hat{x}\hat{x}}(t_o) = 0 \end{aligned} \tag{11}$$

and from [11], the explicit solution is

$$\begin{aligned} &Q_{\hat{x}\hat{x}}(t) = \int_{t_0}^{t}\Phi_{\hat{x}\hat{x}}(t,\tau)K(\tau)V(\tau)K^T(\tau)\Phi_{\hat{x}\hat{x}}^T(t,\tau)\,\mathrm{d}\tau \ge 0 \\ &\dot{\Phi}_{\hat{x}\hat{x}}(t,t_0) = [A - BF(t)]\Phi_{\hat{x}\hat{x}}(t,t_0), \qquad \Phi_{\hat{x}\hat{x}}(t_0,t_0) = I \end{aligned} \tag{12}$$

Also, for Eq. (5), the explicit solution is [16]

$$\begin{aligned} &Q(t) = \bar{Q} + \Phi_e(t,t_0)(Q_0 - \bar{Q}) \\ &\quad \times\left[I + \int_{t_0}^{t}\Phi_e^T(\tau,t_0)C^TV^{-1}C\Phi_e(\tau,t_0)(Q_0 - \bar{Q})\,\mathrm{d}\tau\right]^{-1}\Phi_e^T(t,t_0) \\ &\dot{\Phi}_e(t,t_0) = [A - \bar{Q}C^TV^{-1}C]\Phi_e(t,t_0),\ \Phi_e(t_0,t_0) = I \\ &0 = A\bar{Q} + \bar{Q}A^T + \Gamma W\Gamma^T - \bar{Q}C^TV^{-1}C\bar{Q} \end{aligned} \tag{13}$$

The solution [Eq. (13)] can be written as well as

$$\begin{aligned} &Q(t) = \bar{Q} + \Phi_e(t,t_0)(Q_0 - \bar{Q})^{1/2} \\ &\quad \times\left[I + \int_{t_0}^{t}(Q_0 - \bar{Q})^{T/2}\Phi_e^T(\tau,t_0)C^TV^{-1}C\Phi_e(\tau,t_0)(Q_0 - \bar{Q})^{1/2}\,\mathrm{d}\tau\right]^{-1} \\ &\quad \times(Q_0 - \bar{Q})^{T/2}\Phi_e^T(t,t_0) \end{aligned} \tag{14}$$

Now, if 1) $Q_0 \ge \bar{Q}$ or 2) the terminal phase period $t_f - t_0$ is sufficiently long, the Kalman filter reaches steady state

$$\lim_{t\to\infty} Q(t) = \bar{Q}$$

Assuming $E[x_o] = \bar{x}_o = 0$, we have, subsequently, from Eq. (9),

$$\overline{x^T(t)Gx(t)} = \mathrm{trace}\{G[Q_{\hat{x}\hat{x}} + Q(t)]\} \geq \mathrm{trace}[GQ(t)] \geq \mathrm{trace}[G\bar{Q}] \tag{15}$$

and the GLB we use here is

$$\overline{x^T(t)Gx(t)} \geq \mathrm{trace}[G\bar{Q}] \tag{16}$$

B. Optimal Solution

1. Optimal Guidance Law

The optimal deterministic control (the guidance law) is from the family of proportional navigation [3,11]. The deterministic performance of the modern guidance laws is intensively covered in the literature. The result from these deterministic analyses is that, in absence of guidance noises for sufficiently large acceleration constraint, the miss distance is very small; see the formal proof in Appendix B.

It is assumed that the measurements of the missile states are noise free; thus, guidance law applied with the missile state is implemented directly on the measurements (no estimator is needed).

Thus, based on the preceding observation experience, the dominant contributors to the miss are the guidance noises, which justify the use of the bound in Eq. (16).

Moreover, the guidance law has inherent robustness properties, as shown in [17]. This is due to the fact that the guidance gains approach high values close to intercept.

2. Optimal Estimator

In a radar-guided missile, the most dominant noise source that is not under the control of the designer is the glint noise. The glint noise is a target-created noise, and it can be partially influenced by controlling the relative inertial angular rate of the line of sight (LOS), as expressed in the aircraft coordinates (by the designer of the guidance law), and by the radar designer by applying frequency agility [18], which decorrelates the glint noise measurements. Thus, the optimal estimator is the Kalman filter matched to the target maneuver and glint noise.

C. Greatest Lower Bound on Miss

From this discussion and Eq. (9), it follows that the miss is set by the performance of the Kalman filter. As the covariance matrix Q in Eq. (5) is the state estimation covariance, if setting the state associated to the miss as the first state variable x_1, then the Q_{11} term is the covariance of the miss. Thus, the GLB on the rms miss is derived by assuming that $G = \mathrm{diagonal}([g \quad 0 \quad \cdots \quad 0 \quad 0])$, and from Eq. (16),

$$\mathrm{GLB}\ (\mathrm{rms\ miss}) = \sqrt{Q_{11}(t_f)} \tag{17}$$

No other stochastic guidance law (or estimator), subject to the preceding detailed assumptions, can achieve smaller rms miss.

IV. Spectrum of Glint Noise

As shown in [14], the angular noises (range-independent LOS angular noise, passive receiver noise, and active receiver noise) for correctly designed estimator and guidance laws contribute very small (zero) miss distance. The intuitive explanation, beyond the rigorous treatment in [14], for this is the measurement noise component perpendicular to the LOS x due to angular range-dependent and independent noises (with variance of σ) is going to zero as the missile-target range R closes; that is, $x = \sigma R \to 0$ as $R \to 0$. Thus, the contribution of these noises (except the glint noise) on the terminal estimation zero is very small (zero).

Therefore, in this paper, only the glint noise is considered. The standard deviation of the glint noise σ_g for uniformly distributed reflectors is [18]

$$\sigma_g^2 = \frac{1}{12}D^2 \qquad D = \sqrt{\frac{2}{\pi}}L \tag{18}$$

where D is the effective linear dimension of the target perpendicular to the target-missile LOS and L is the linear dimension of the target perpendicular to the target-missile LOS.

In general, the glint noise is correlated. The correlation depends on the relative angular rate between the aircraft and the missile. Application of frequency agility [18] decorrelates the glint measurements. The effect of this decorrelation is that the glint noise energy is spread over a larger bandwidth, thus less energy is concentrated within the relevant bandwidth of the guidance loop; that is, the spectral density of the glint at low frequencies is smaller with respect to the spectral density without frequency agility.

When frequency agility is applied at a rate of $f_s = 1/T_s$ Hz, the spectral density of the glint (a stair-type random stochastic process or independent and identically distributed sequence) is given by [19]

$$V_{go} = \sigma_g^2 T_s\ \mathrm{m^2/Hz} \qquad V_g(\omega) = V_{go}\left[\frac{\sin(\omega T_s/2)}{(\omega T_s/2)}\right]^2\ \mathrm{m^2/Hz} \tag{19}$$

where T_s is the sampling rate of the frequency agile radar (in seconds).

Therefore, the spectral density of the measurement noise $v(t)$ is V_{go} $\mathrm{m^2/Hz}$.

V. Target Maneuver Models: Shaping Filters

In this work, the following target maneuvers from the large family of maneuvers considered in the literature [3,20–27] are considered: 1) abruptly starting sine-wave maneuver, 2) continuous harmonic wave maneuver, and 3) nearly CT maneuver.

Guidance laws for these types of maneuvers are derived in [3,6,7].

VI. Abruptly Starting Harmonic Wave Maneuver

Here, an abruptly starting damped sinusoidal waveform target maneuver is dealt with. This description includes the pure sinusoid for $\zeta = 0$ (no damping) [22,23,25]. The phase φ is arbitrary but not random. Thorough treatment of the estimation of sinusoidal maneuvers with known and unknown frequencies is presented in [28]. The target acceleration of an abruptly starting sinusoid maneuver is described in this work by

$$\ddot{y} = a_T = a_{To}\{e^{-\zeta\omega_{\mathrm{BR}}(t-T)}\sin[\omega_{\mathrm{BR}}\sqrt{1-\zeta^2}(t-T) + \varphi]\}\mu(t-T), \quad 0 \leq \zeta \leq 1 \tag{20}$$

where y_T is the target distance relative to a reference in meters, a_T is the target acceleration in meters per second squared, a_{To} is the target maneuver acceleration level in meters per second squared, ω_{BR} is a frequency of the sinusoidal (barrel roll) maneuver in radians per second, ζ is the damping factor of the barrel-roll-type target maneuver, φ is the initial phase (in radians), and $\mu(t)$ is the step function.

The preceding assumes that the target performs an evasive maneuver (a stochastic process); that is, a sinusoidal/barrel-roll-type acceleration maneuver of amplitude a_{To}, for which the initiation instant T is uniformly distributed in the interval $[t_o, t_f]$. The shaping filter [22–26] of this process is represented by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} a_T(t) \\ \eta_T(t) \end{bmatrix} = \begin{bmatrix} -2\zeta\omega_{\mathrm{BR}} & 1 \\ -\omega_{\mathrm{BR}}^2 & 0 \end{bmatrix}\begin{bmatrix} a_T(t) \\ \eta_T(t) \end{bmatrix} + \begin{bmatrix} \sin\varphi \\ \sqrt{1-\zeta^2}\omega_{\mathrm{BR}}\cos\varphi + \zeta\omega_{\mathrm{BR}}\sin\varphi \end{bmatrix} w_{\mathrm{BR}}(t) \tag{21}$$

that is represented in the Laplace domain as

$$\frac{a_T(s)}{w_{\mathrm{BR}}(s)} = \frac{s+\varsigma\omega_{\mathrm{BR}}}{s^2+2\zeta\omega_{\mathrm{BR}}s+\omega_{\mathrm{BR}}^2}\sin\varphi + \frac{\sqrt{1-\varsigma^2}\omega_{\mathrm{BR}}}{s^2+2\zeta\omega_{\mathrm{BR}}s+\omega_{\mathrm{BR}}^2}\cos\varphi \tag{22}$$

where the spectral density of the target maneuver (the process noise) $w_{\mathrm{BR}}(t)$ is

$$W_{\mathrm{BR}} = \frac{a_{To}^2}{t_m}\ (\mathrm{m/s^3})^2/\mathrm{Hz} = \mathrm{m^2/s^5} \tag{23}$$

where $t_m = t_f - t_o$.

This type of target acceleration buildup can be described by the following dynamic model of the target-missile encounter:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2\zeta\omega_{\mathrm{BR}} & 1 \\ 0 & 0 & -\omega_{\mathrm{BR}}^2 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ \sin\varphi \\ (\sqrt{1-\zeta^2}\cos\varphi + \zeta\sin\varphi)\omega_{\mathrm{BR}} \end{bmatrix} w_{\mathrm{BR}}(t)$$

$$z(t) = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ \eta(t) \end{bmatrix} + v(t) \tag{24}$$

where y—target–missile separation distance ($y = y_T - y_M$) in meters, $\dot{y}$ is the target-missile separation velocity in meters per second, and a_M is the missile acceleration in meters per second squared.

The spectral density of the measurement noise $v(t)$ is V_{go} $\mathrm{m^2/Hz}$.

The following deals with the special case, abruptly starting sine wave: $\zeta = 0$ and $\varphi = 0$. The maneuver is a sine wave that initiates $t_f - T$ seconds before intercept:

$$\ddot{y}_T = a_T = a_{To}\sin[\omega_{\mathrm{BR}}(t-T)]\mu(t-T) \tag{25}$$

then

$$\ddot{a}_T + \omega_{\mathrm{BR}}^2 a_T = \omega_{\mathrm{BR}} w_{\mathrm{BR}}(t) = w'_{\mathrm{BR}}(t) \tag{26}$$

The resulting shaping filter [22,23,26] is

$$\frac{a_T(s)}{w_{\mathrm{BR}}(s)} = \frac{\omega_{\mathrm{BR}}}{s^2+\omega_{\mathrm{BR}}^2} \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} a_T(t) \\ j_T(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_{\mathrm{BR}}^2 & 0 \end{bmatrix}\begin{bmatrix} a_T(t) \\ j_T(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w'_{\mathrm{BR}}(t) \tag{28}$$

where the spectral density of the target maneuver (the process noise) $w'_{\mathrm{BR}}(t)$ is

$$W'_{\mathrm{BR}} = \omega_{\mathrm{BR}}^2\frac{a_{To}^2}{t_m} = \frac{j_{To}^2}{t_m}\ (\mathrm{m/s^4})^2/\mathrm{Hz} = \mathrm{m^2/s^7} \tag{29}$$

where j_{To} denotes the jerk.

In this case, the fourth state can be interpreted as the jerk. The state-space representation of the abruptly starting sine-wave maneuver is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{\mathrm{BR}}^2 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w'_{\mathrm{BR}}(t)$$

$$z(t) = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} + v(t) \tag{30}$$

The rms miss distance with frequency agility for the abruptly starting sine-wave maneuver is

$$\bar{Q} = V_{go}\begin{bmatrix} 2.61\omega_{o_{\sin}e}\kappa_{\mathrm{BR}\sin e}^2 & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$\omega_{o_{\mathrm{sine}}} = \sqrt[8]{\frac{W'_{\mathrm{BR}}}{V_{go}}} \qquad \kappa_{\mathrm{BRsine}} = \kappa_{\mathrm{BRsine}}(\omega_{o_{\mathrm{sine}}}/\omega_{\mathrm{BR}}) \tag{31}$$

and the GLB of the rms miss is from Eq. (17):

$$\text{rms miss m} \geq 0.54\kappa_{\mathrm{BRsine}}(\omega_{o_{\mathrm{sine}}}/\omega_{\mathrm{BR}})\sqrt[16]{\frac{T_s^7 D^{14}\omega_{\mathrm{BR}}^2 a_T^2}{t_m}} \tag{32}$$

Figure 1 presents the correction factor $\kappa_{\mathrm{BR}\sin e}(\omega_{o_{\mathrm{sine}}}/\omega_{\mathrm{BR}})$ for the GLB of the rms miss distance for the abruptly starting sine-wave acceleration maneuver versus $\omega_{o_{\mathrm{sine}}}/\omega_{\mathrm{BR}}$.

Remark: This type of normalization [29] had been selected as for the case of $\omega_{\mathrm{BR}} \to 0$, the case of abruptly starting sine-wave maneuver converges to the constant jerk maneuver, for which the solution is presented in [5].

VII. Continuous Barrel-Roll Maneuver

This section presents a type of sinusoidal maneuver that has not been treated in the literature. In contrast with the previous section,

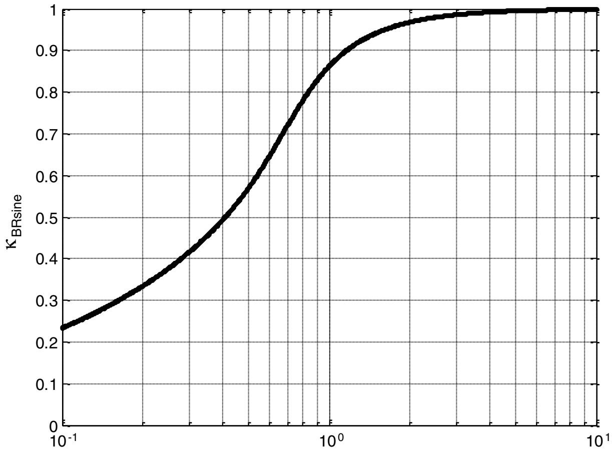

Fig. 1 Correction factor of rms miss distance GLB for abruptly starting sine-wave target maneuver.

here, a target barrel-roll (sinusoidal) maneuver starts before the track initiation and continues until intercept.

The previous models of harmonic target maneuver, as presented in the preceding section, assume the following [3,20–31]:

1) The frequency is known exactly and constant.

2) The maneuver starts abruptly.

3) The initiation instant is uniformly distributed in some interval $U[0, t_m]$.

The drawbacks of these treatments are as follows:

1) The frequency is never known exactly.

2) The frequency is not constant.

3) The initiation instant distribution cannot be systematically rationalized: heuristics are needed.

In the target maneuver model presented in this section, the following is applicable:

1) For realistic target maneuvers, the new model gives better representation.

2) Initiation of tracking is uniformly distributed over the sine-wave period, thus avoiding heuristics.

3) The new model describes the uncertainty of the maneuver frequency by its distribution.

4) The new model models sinusoidal maneuver with constant random frequency.

5) The new model models spiraling maneuver [6,31] with time-varying frequency (swept sine [32]).

6) Models a narrow band noiselike maneuver (randomly changing frequency [33]).

A. Spectrum of Random Frequency Sinusoidal Waveform

This section uses the result [19] that the continuous sinusoidal wave stochastic process $\xi(t)$, with random frequency ω_T, random phase φ, and random amplitude ξ_o, defined as

$$\begin{gathered}\xi(t) = \xi_o \sin(\omega_T t - \varphi) \qquad p(\xi_o, \omega_T, \varphi) = p_{\xi_o}(x) p_{\omega_T}(\omega) p_\varphi(\phi) \\ \omega_T \approx p_{\omega_T}(\omega) = \frac{1}{a} S_\xi(\omega); \qquad a = \int_{-\infty}^{\infty} S_\xi(\omega)\, \mathrm{d}\omega \\ p_{\xi_o}(\xi) \text{ is known } \varphi \approx U[0, 2\pi]\end{gathered} \tag{33}$$

has the spectrum $S_\xi(\omega)$. In other words, if the frequency of the sine wave ω_T is distributed $S_\xi(\omega)$, then the spectrum of $\xi(t)$ is $S_\xi(\omega)$ (the spectrum of ξ takes the shape of the probability distribution function of the random frequency ω_T).

Specifically, if the following represents the distribution of the frequency of the sine-wave maneuver

$$S_\xi(\omega) = H_\xi(s) H_\xi(-s)|_{s=j\omega} \tag{34}$$

then the process $\tilde{\xi}(t)$ that is created by passing white Gaussian noise of the correct power spectral density though a filter $[H_\xi(s)]$ is indistinguishable from $\xi(t)$ up to the second moment; that is, $E[\xi(t)] = E[\tilde{\xi}(t)] = 0$ and $E[\xi^2(t)] = E[\tilde{\xi}^2(t)]$. This is exactly the same concept used in derivation of the shaping filter [22,23,25].

B. Shaping Filter of Continuous Random Frequency Sinusoidal Maneuver

Here, the spectrum of the target continuous random frequency sinusoidal maneuver [continuous barrel-roll (CBR) maneuver] is approximated by a rational function:

$$H_\xi(s) = \frac{\xi(s)}{w_\xi(s)} = \frac{s}{s^2 + 2\zeta\omega_\xi s + \omega_\xi^2} \tag{35}$$

where $\omega_\xi = \omega_{\mathrm{CBR}}$ is the center of the frequency distribution. Then,

$$\sigma_\xi^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} H_\xi(s) H_\xi(-s) W_\xi \, \mathrm{d}s \tag{36}$$

where W_ξ is the power spectral density of system driving noise $w_\xi(t)$, and we have [34]

$$\sigma_\xi^2 = \frac{W_\xi}{4\zeta\omega_\xi} = \frac{W_\xi}{2\Delta\omega_\xi} \tag{37}$$

where $\Delta\omega_\xi$ is the spread of the distribution (the barrel-roll frequency uncertainty).

The rms of the sine-wave stochastic process in Eq. (33) with amplitude ξ_o [distributed as $p_\xi(x)$ with zero mean and variance of $\bar{\xi}_o^2$] is

$$\sigma_\xi^2 = \tfrac{1}{2}\bar{\xi}_o^2 \tag{38}$$

Equating Eqs. (37) and (38), we get

$$\zeta = \frac{1}{2} \frac{\Delta\omega_\xi}{\omega_\xi} \tag{39}$$

$$W_\xi = 2\zeta\omega_\xi \bar{\xi}_o^2 = \Delta\omega_\xi \bar{\xi}_o^2 \tag{40}$$

A state-space representation of Eq. (35) is

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} -2\zeta\omega_\xi & 1 \\ -\omega_\xi^2 & 0 \end{bmatrix}\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_\xi(t) \\ \xi(t) = [1 \quad 0]\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}\end{gathered} \tag{41}$$

C. Random Frequency Continuous Sinusoidal Maneuver

The state-space representation of the continuous sinusoidal (barrel roll) target acceleration maneuver is

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2\zeta\omega_{\mathrm{CBR}} & 1 \\ 0 & 0 & -\omega_{\mathrm{CBR}}^2 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ \eta(t) \end{bmatrix} \\ + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w_{\mathrm{CBR}}(t) \\ z(t) = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ \eta(t) \end{bmatrix} + v(t)\end{gathered} \tag{42}$$

where $\eta(t)$ is a state variable of the sinusoidal maneuver representation.

The spectral density of the target maneuver (the process noise) $w_{\mathrm{CBR}}(t)$ is

$$W_{\mathrm{CBR}} = 2\zeta\omega_{\mathrm{CBR}}\bar{a}_{To}^2 = \Delta\omega_{\mathrm{CBR}}\bar{a}_{To}^2 \ (\mathrm{m/s^3})^2/\mathrm{Hz} \tag{43}$$

This representation is not minimal; specifically, it is not controllable with respect to the process noise input. A minimal representation of Eq. (42) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_{\mathrm{CBR}}^2 & -2\zeta\omega_{\mathrm{CBR}} \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_{\mathrm{CCBR}}$$

$$z(t) = [1 \quad 0 \quad 0]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \end{bmatrix} + v(t) \tag{44}$$

This representation is used in the following derivations.

D. Root-Mean-Sqare Tracking Error Distance with Frequency Agility

The rms tracking error distance with frequency agility for continuous sinusoidal (barrel roll) target maneuver is

$$\bar{Q} = V_{go}\begin{bmatrix} 2\omega_{o_{\mathrm{CBR}}}\kappa_{\mathrm{CBR}}^2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \qquad \omega_{o_{\mathrm{CBR}}} = \sqrt[6]{\frac{W_{\mathrm{CBR}}}{V_{go}}}$$

$$\kappa_{\mathrm{CBR}} = \kappa_{\mathrm{CBR}}(\omega_{o_{\mathrm{CBR}}}/\omega_{\mathrm{CBR}}) \tag{45}$$

and the GLB of the rms miss from Eq. (17) is

$$\text{rms miss m} \geq 0.40\kappa_{\mathrm{CBR}}(\omega_{o_{\mathrm{CBR}}}/\omega_{\mathrm{CBR}})\sqrt[12]{T_s^5 D^{10}\zeta\omega_{\mathrm{CBR}} a_{To}^2}$$
$$\geq 0.38\kappa_{\mathrm{CBR}}(\omega_{o_{\mathrm{CBR}}}/\omega_{\mathrm{CBR}})\sqrt[12]{T_s^5 D^{10}\Delta\omega_{\mathrm{CBR}} a_{To}^2} \tag{46}$$

Figure 2 presents the correction factor $\kappa_{\mathrm{CBR}}(\omega_{o_{\mathrm{CBR}}}/\omega_{\mathrm{CBR}})$ for the GLB of the rms miss distance of continuous sinusoidal target maneuver versus $\omega_{o_{\mathrm{CBR}}}/\omega_{\mathrm{CBR}}$.

Notice that, for $\zeta = 0$, if the track is initiated soon enough (the transient died out), then very small rms miss distance is achievable.

Remark: This type of normalization [Eq. (45)] had been selected as for the case of $\omega_{\mathrm{CBR}} \to 0$; the case of continuous sinusoidal target maneuver converges to the constant acceleration (CA) maneuver, for which solution is presented in [5].

VIII. Constant Turn-Rate Maneuver

Here, a target maneuver that is called a nearly CT maneuver is dealt with. This description includes the fully constant-turn (CT) maneuver. An exhaustive survey of turn-rate models is presented in

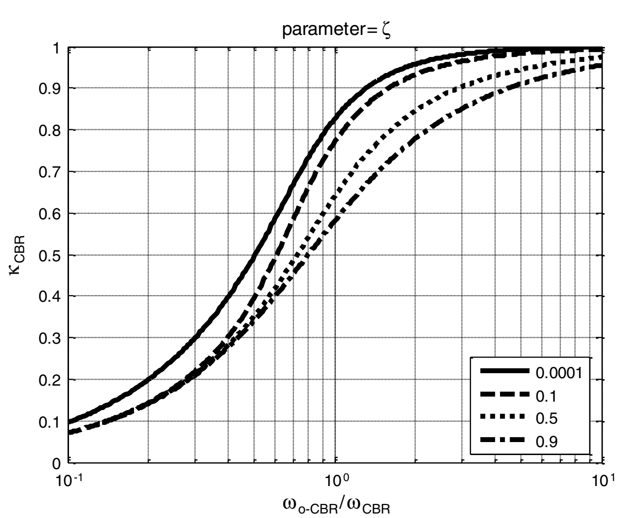

Fig. 2 Correction factor for rms miss distance GLB for continuous sinusoidal (barrel roll) target maneuver.

[25]. The specific target maneuver model used here has been introduced in [27].

The state-space representation of a nearly CT target maneuver for each Cartesian coordinate, x, y, or z, [25] is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_{\mathrm{CT}}^2 & -2\zeta\omega_{\mathrm{CT}} \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_{\mathrm{CT}}$$

$$z(t) = [1 \quad 0 \quad 0]\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} + v(t) \tag{47}$$

The CT is

$$\omega_{\mathrm{CT}} = \frac{a_{To}}{V_T} \ \text{rad/s} \tag{48}$$

where V_T is the target velocity in meters per second, a_T is the target acceleration in meters per seconds squared, and ζ is a measure of the deviation of the maneuver from exactly CT [35].

We have

$$H_{\mathrm{CT}}(s) = \frac{\ddot{x}}{w_{\mathrm{CT}}} = \frac{s}{s^2 + 2\zeta\omega_{\mathrm{CT}}s + \omega_{\mathrm{CT}}^2} \tag{49}$$

and

$$\sigma_{\ddot{x}}^2 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_{\mathrm{CT}}(s)H_{\mathrm{CT}}(-s)W_{\mathrm{CT}}\,\mathrm{d}s \tag{50}$$

where W_{CT} is the power spectral density of system driving noise $w_{\mathrm{CT}}(t)$, and from Eq. (37),

$$\sigma_{\ddot{x}}^2 = \frac{W_{\mathrm{CT}}}{4\zeta\omega_{\mathrm{CT}}} = \frac{W_{\mathrm{CT}}}{2\Delta\omega_{\mathrm{CT}}} \tag{51}$$

where $\Delta\omega_{\mathrm{CT}}$ is the spread of the distribution (the angular turn-rate uncertainty).

The projection of the coordinated turn-rate acceleration on one of the Cartesian coordinates is a sine wave with amplitude a_{To}. The rms of the projection of the acceleration is

$$\sigma_{\ddot{x}}^2 = \tfrac{1}{2}\bar{a}_{To}^2 \tag{52}$$

Equating Eqs. (51) and (52) gives

$$\varsigma = \frac{1}{2}\frac{\Delta\omega_{\mathrm{CT}}}{\omega_{\mathrm{CT}}} \tag{53}$$

and

$$W_{\mathrm{CT}} = 2\varsigma\omega_{\mathrm{CT}}\bar{a}_{To}^2 = \Delta\omega_{\mathrm{CT}}\bar{a}_{To}^2 \ (\mathrm{m/s^3})^2/\mathrm{Hz} = \mathrm{m^2/s^5} \tag{54}$$

The spectral density of the measurement noise $v(t)$ is V_{go} [in meters squared per Hertz).

The rms miss distance with frequency agility for a nearly CT target maneuver is

$$\bar{Q} = V_{go}\begin{bmatrix} 2\omega_{o_{\mathrm{CT}}}\kappa_{\mathrm{CT}}^2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \qquad \omega_{o_{\mathrm{CT}}} = \sqrt[6]{\frac{W_{\mathrm{CT}}}{V_{go}}}$$

$$\kappa_{\mathrm{CT}} = \kappa_{\mathrm{CT}}(\omega_{o_{\mathrm{CT}}}/\omega_{\mathrm{CT}}) \tag{55}$$

and the GLB of the rms miss for each Cartesian coordinate is from Eq. (17):

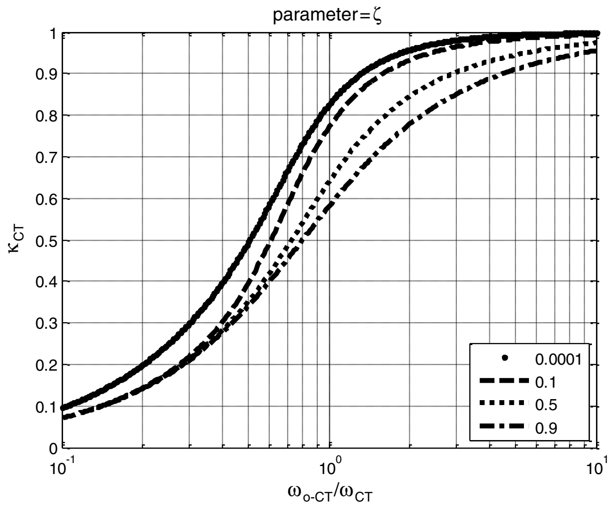

Fig. 3 Correction factor of rms miss distance GLB for nearly CT target maneuver.

$$\text{rms miss m} \geq 0.40\kappa_{\text{CT}}(\omega_{o_{\text{CT}}}/\omega_{\text{CT}})\sqrt[12]{T_s^5 D^{10}\zeta\omega_{\text{CT}} a_{To}^2}$$
$$\geq 0.40\kappa_{\text{CT}}(\omega_{o_{\text{CT}}}/\omega_{\text{CT}})\sqrt[12]{\frac{T_s^5 D^{10}\zeta a_{To}^3}{V_T}} \tag{56}$$

Figure 3 presents the correction factor $\kappa_{\text{CT}}(\omega_{o_{\text{CT}}}/\omega_{\text{CT}})$ for the GLB of the rms miss distance for the CT maneuver versus $\omega_{o_{\text{CT}}}/\omega_{\text{CT}}$.

Remark 1: This type of normalization [Eq. (55)] had been selected as for the case of $\omega_{\text{CT}} \to 0$; the case of CT target maneuver converges to the CA maneuver, for which the solution is presented in [5].

Remark 2: Although the CT and the random CBR target maneuver models had been derived from different approaches, they result in the same structure of the solution. Both are presented here to enable the system engineer to differentiate these two target maneuver models.

IX. Example

For the examples here, it is assumed

$$V_{go} = 1\ \text{m}^2/\text{Hz} \qquad a_{To} = 50\ \text{m}^2/\text{s} \qquad V_T = 500\ \text{m/s}$$
$$t_m = 5\ \text{s} \qquad \omega_{\text{BR}} = \pi\ \text{rad/s}\ (f_{\text{CBR}} = 0.5\ \text{Hz})$$
$$\Delta\omega_{\text{CBR}} = \pi/10\ \text{rad/s}$$
$$\Delta f_{\text{CBR}} = \pm 0.025\ \text{Hz}(10\%\ \text{uncertainty})$$

A. Abruptly Starting Sine Wave: $\zeta = 0$ and $\varphi = 0$

For the abruptly starting sine wave, the formula is

$$\text{GLB (rms miss)} = \sqrt{Q_{11}} = \sqrt{V_{go} 2.61 \kappa_{\text{BR}\sin e}^2 \omega_{o_{\sin e}}}$$

Then

$$\omega_{o_{\sin e}} = \sqrt[8]{\frac{W'_{\text{BR}}}{V_{go}}} = \sqrt[8]{\frac{\omega_{\text{BR}}^2(a_{To}^2/t_m)}{V_{go}}} = \sqrt[8]{\frac{\pi^2(50^2/5)}{1}} = 2.9\ \text{rad/s};$$
$$\frac{\omega_{o_{\sin e}}}{\omega_{\text{BR}}} = \frac{2.9}{\pi} = 0.92 \qquad \kappa_{\text{BR}\sin e} = 0.85$$
$$\text{GLB(rms miss)} = \sqrt{2.61}\cdot 0.85 \cdot \sqrt{2.9} = 2.3\ \text{m}$$

B. Continuous Barrel-Roll Target Maneuver

For the CBR maneuver, the formula is

$$\text{GLB (rms miss)} = \sqrt{Q_{11}} = \sqrt{V_{go} 2\kappa_{\text{CBR}}^2 \omega_{o_{\text{CBR}}}}$$

Then

$$\omega_{o_{\text{CBR}}} = \sqrt[6]{\frac{W_{\text{CBR}}}{V_{go}}} = \sqrt[6]{\frac{\Delta\omega_{\text{CBR}}\bar{a}_{To}^2}{V_{go}}} = \sqrt[6]{\frac{\pi/10\cdot 50^2}{1}} = 3.03$$
$$\zeta = \frac{1}{2}\frac{\Delta\omega_{\text{CBR}}}{\omega_{\text{CBR}}} = \frac{1}{2}\frac{\pi/10}{\pi} = \frac{1}{20} \qquad \frac{\omega_{o_{\text{CBR}}}}{\omega_{\text{BR}}} = \frac{3.03}{\pi} = 0.96$$
$$\kappa_{\text{CBR}} = 0.8 \qquad \text{GLB(rms miss)} = \sqrt{2}\cdot 0.8 \cdot \sqrt{3.03} = 1.96\ \text{m}$$

C. Constant Turn-Rate Target Maneuver

For the CT maneuver, the formula is

$$\text{GLB (rms miss)} = \sqrt{Q_{11}} = \sqrt{V_{go} 2\kappa_{\text{CT}}^2 \omega_{o_{\text{CT}}}}$$

Then

$$\omega_{\text{CT}} = \frac{a_{To}}{V_T} = \frac{50}{500} = 0.1(0.016\ \text{Hz})$$
$$\Delta\omega_{\text{CT}} = 0.01(10\%\ \text{uncertainty})$$
$$\omega_{o_{\text{CT}}} = \sqrt[6]{\frac{W_{\text{CT}}}{V_{go}}} = \sqrt[6]{\frac{\Delta\omega_{\text{CT}}\bar{a}_{To}^2}{V_{go}}} = \sqrt[6]{\frac{0.01\cdot 50^2}{1}} = 1.7$$
$$\zeta = \frac{1}{2}\frac{\Delta\omega_{\text{CT}}}{\omega_{\text{CT}}} = \frac{1}{2}\frac{0.01}{0.1} = \frac{1}{20}\ \frac{\omega_{o_{\text{CT}}}}{\omega_{\text{CT}}} = \frac{1.7}{0.1} = 17 \qquad \kappa_{\text{CBR}} = 0.92$$
$$\text{GLB(rms miss)} = \sqrt{2}\cdot 0.92 \cdot \sqrt{1.7} = 1.7\ \text{m}$$

X. Conclusions

Explicit formulas of the GLB on the rms miss distance of a radar-guided missile against harmonic-sinusoidal maneuvers are presented. The GLB depends on a small set of core parameters: the size, the acceleration level, the velocity of the target, the period of the encounter, and the data rate. The formulas can be used for preliminary assessment of the missile performance and for validation of simulation performance.

Appendix A: Derivations

We wish to derive an expression for the difference in performance due to the use of an estimator (required due to noisy measurements).

The system is

$$\dot{x}(t) = Ax(t) + Bu(t) + \Gamma w(t), \qquad x(t_o) = x_o$$
$$z(t) = Cx(t) + v(t) \tag{A1}$$

The estimator is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(t)[z(t) - C\hat{x}(t)], \qquad \hat{x}(t_o) = \bar{x}_o \tag{A2}$$

The feedback for full-state noiseless measurements system is

$$u(t) = -F(t)x(t) \tag{A3}$$

and for noisy measurements (stochastic system), it is

$$u(t) = -F(t)\hat{x}(t) \tag{A4}$$

Remarks on the approach to the derivations are as follows:

1) From the target point of view, its behavior is completely known, i.e., deterministic. That is, $w(t)$ is known to the target.

2) However, from the point of view of the missile, $w(t)$ is unknown; that is, the best the missile can do is to assume it is random. The Kalman filter, together with the shaping filter, provides a framework to build estimators for which a statistical description of $w(t)$ is sufficient (at least the first two moments: mean and correlation-spectral density). This is what the shaping filter is used for (the design of the Kalman filter). Once these statistical descriptors are used for the Kalman filter design, they are used no more in the analytical analysis.

3) In a real encounter, the target picks up one sample from the maneuver ensemble. This is then replicated by the Monte Carlo simulation.

4) The following analysis realizes this and uses it for the derivation of the bound.

The closed-loop system for full-state noiseless measurements (deterministic) is

$$\begin{aligned}\dot{x}_d(t) &= Ax_d(t) - BFx_d(t) + \Gamma w(t), \qquad x_d(t_o) = x_o\\ &= (A - BF)x_d(t) + \Gamma w(t)\end{aligned} \tag{A5}$$

The closed-loop system for full-state feedback control with noisy measurements (stochastic) is

$$\begin{aligned}\dot{x}_s(t) &= Ax_s(t) - BF\hat{x}_s(t) + \Gamma w(t), \qquad x_s(t_o) = x_o\\ &= Ax_s(t) - BF(\hat{x}_s(t) - e) + \Gamma w(t)\\ &= (A - BF)x_s(t) + BFe + \Gamma w(t)\end{aligned} \tag{A6}$$

The difference between the two is

$$x_{sd}(t) = x_s(t) - x_d(t) \tag{A7}$$

and

$$\begin{aligned}\dot{x}_{sd} &= (A - BF)x_s + BFe + \Gamma w - [(A - BF)x_d + \Gamma w]\\ &= (A - BF)x_{sd} + BFe\\ x_{sd}(t_o) &= x_{sdo} = x_s(t_o) - x_d(t_o) = x_o - x_o = 0\end{aligned} \tag{A8}$$

where

$$\begin{aligned}\dot{e}(t) &= (A - KC)e + \Gamma w - Kv, \qquad e(t_o) = e_o = x_o - \hat{x}(t_o) = x_o\\ \dot{\Phi}_e(t, t_o) &= (A - KC)\Phi_e(t, t_o), \qquad \Phi_e(t_o, t_o) = I\end{aligned} \tag{A9}$$

In Eq. (A8), the fact that the target uses one sample from the ensemble as its maneuver is used. This shows that the miss is caused by the fact that the Kalman filter of the missile is matched to the ensemble of the maneuvers and not to the specific maneuver (the maneuver that the target currently performs). If the missile knew the target maneuver, the steady-state estimation error $e(t)$ would have been zero.

This type of analysis gives us the tool to separately see the performance $x_d(t)$ with perfect measurements (this is one type of bound) and the contribution of the noisy measurements $x_{sd}(t)$ (other type of bound). We use the linearity of the system to apply the principle of superposition to analyze the contribution of the individual miss contributors.

We then have the following.

I. Influence of Heading Error: x_o Only

We have

$$\begin{aligned}\dot{e}(t) &= (A - KC)e, \qquad e_o = x_o, \qquad e(t) = \Phi_e(t, t_o)x_o\\ \dot{x}_{sd} &= (A - BF)x_{sd} + BF\Phi_e(t, t_o)x_o, \qquad x_{sdo} = 0\\ x_{sd}(t) &= \int_{t_o}^{t} \Phi_{sd}(t, \tau)BF(\tau)\Phi_e(\tau, t_o)x_o\, \mathrm{d}\tau\\ \dot{\Phi}_{sd}(t, t_o) &= [A - BF(t)]\Phi_{sd}(t, t_o)\Phi_{sd}, \qquad (t_o, t_o) = I\end{aligned} \tag{A10}$$

Furthermore, we write

$$e(t_1) = \Phi_e(t_1, t_o)x_o \qquad e(t) = \Phi_e(t, t_1)e(t_1) \tag{A11}$$

then

$$\begin{aligned}x_{sd}(t_1) &= \int_{t_o}^{t_1} \Phi_{\xi}(t_1, \tau)BF(\tau)\Phi_e(\tau, t_o)x_o\, \mathrm{d}\tau\\ x_{sd}(t) &= \Phi_{sd}(t, t_1)x_{sd}(t_1)\\ &\quad + \int_{t_1}^{t} \Phi_{sd}(t, \tau)BF(\tau)\Phi_e(\tau, t_1)e(t_1)\, \mathrm{d}\tau\end{aligned} \tag{A12}$$

From here, we can get an estimate on the transient time of the estimator for reducing the heading error. The required time t_1 is given by

$$e(t_1) = \Phi_e(t_1, t_o)x_o \tag{A13}$$

For the target acceleration step maneuver, this has been solved in [16]. One can see that $\Phi_e(\,)$ decays, to the steady state of the Riccati equation, at a rate of $2/\omega_o$:

$$\|\Phi_e(t, 0)\| \leq \exp\left(-\frac{\omega_o t}{2}\right), \quad \text{where } \omega_o = \sqrt[6]{\frac{W}{V}} \tag{A14}$$

II. Influence of Target Maneuver Only

By the superposition assumption, there is no measurement noise; that is, $v(t) = 0$. The maneuver is known to the designer of the guidance law (see Sec. V), and the guidance law is fully matched to this maneuver. There is no heading error (superposition assumed); therefore, the estimation error $e(t)$ is bounded. The closed-loop system for full-state noisy measurements is

$$\begin{aligned}\dot{e}(t) &= [A - K(t)C]e(t) + \Gamma w(t), \qquad e(t_o) = 0\\ x(t) &= Ax(t) + Bu(t) + \Gamma w(t) + BFe(t), \qquad x(t_o) = x_o\end{aligned} \tag{A15}$$

In the derivation (the guidance law), the model taken into account is

$$\begin{aligned}x(t) &= Ax(t) + Bu(t) + \Gamma w(t), \qquad x(t_o) = x_o\\ w(t) &= a_{To}S(t - T)\end{aligned} \tag{A16}$$

where $S(\,)$ represents one of the sine-wave maneuvers. Thus, $w(t)$ is a stochastic process, as T is a random variable, $T \cong U[0, t_m]$.

The guidance law is then Eq. (6), and the term

$$\begin{aligned}BFe(t) &= BF(t)\int_{t}^{t_f} \Phi_{ee}(t_f, \tau)\Gamma w(\tau)\, \mathrm{d}\tau\\ \dot{\Phi}_{ee}(t, t_o) &= (A - K(t)C)\Phi_{ee}(t, t_o), \qquad \dot{\Phi}_{ee}(t_o, t_o) = I\end{aligned} \tag{A17}$$

is not accounted for in the guidance law when considering the ensemble average. Through this term, the dynamics of the estimator and the target maneuver contribute to the miss (in addition to the measurement noise as computed in the paper), as the guidance law does not compensate for it.

In the computation, the guidance law takes into account the model

$$\begin{aligned}x(t) &= Ax(t) + Bu(t) + \Gamma w(t), \qquad x(t_o) = x_o\\ E[w(t)w^T(t)] &= W\delta(t - T)\end{aligned} \tag{A18}$$

where W is the spectral density of the white noise Gaussian stochastic process. The shaping filter formalism takes care that the first two moments of the system driving processes in Eqs. (A16) and (A18) are the same [e.g., Eqs. (23), (29), and (49)].

For a specific single sample of the acceleration profile, the performance will be governed by

$$x(t) = Ax(t) + Bu(t) + BFe(t), \qquad x(t_o) = x_o \tag{A19}$$

For $e(t) = 0$, we get $x(t_f) = 0$. However, for $e(t_0) \neq 0$, the terminal miss is given by

$$x(t) = [A - BF(t)]x(t) + BF(t)\left[\int_t^{t_f} \Phi_{ee}(t_f, \tau)\Gamma w(\tau)\,\mathrm{d}\tau\right]e(t_o) \tag{A20}$$

Without the last term, the miss would have been zero (for $g \to 0$), as detailed in Appendix B.

Appendix B: Zero Miss Distance

The control objective in Eq. (4), following [11], Chap. 14, is

$$J = \frac{1}{2}E\left[\hat{x}^T(t_f)G\hat{x}(t_f) + \int_{t_o}^{t_f}[\hat{z}^T(t)Q_c\hat{z}(t) + u^T(t)Ru(t)]\,\mathrm{d}t\right] + E\left[e^T(t_f)Ge(t_f) + \int_{t_o}^{t_f} e^T(t)C^TQ_cCe(t)\,\mathrm{d}t\right] \tag{B1}$$

This is due to the separation theorem and certainty equivalence principle. The guidance law can only effect the first term: the control term. The second term is minimized solely by the Kalman filter. We have [[14], theorem 3.4] that, when the optimal control is applied, then

$$J = \frac{1}{2}\left[\hat{x}^{oT}(t_f)G\hat{x}^o(t_f) + \int_{t_o}^{t_f}[\hat{x}^{oT}(t)C^TQ_cC\hat{x}^o(t) + u^{oT}(t)Ru^o(t)]\,\mathrm{d}t\right] = \hat{x}^{oT}(t_o)P(t_o)\hat{x}^o(t_o) \tag{B2}$$

Moreover, when the system is stabilizable (controllable) and reconstructible (observable), then $P(t_o)$ is bounded ([15], theorem 3.5). So, $\hat{x}^{oT}(t_o)P(t_o)\hat{x}^o(t_o)$ is finite. Now, if some diagonal component of G, i.e., $g_{ii} \to \infty$, then, necessarily, the respective component of $\hat{x}^o(t_f)$ (i.e., $\hat{x}_i^o(t_f) \to 0$ (at least as fast as $\sqrt{g_{ii}}$); see [17].

This means that the missile is guided exactly to wherever the estimator is pointing at and, consequently, the actual miss is the estimation error.

Appendix C: Derivation of Equation (32)

In [5], the analytic solution of the steady-state Riccati equation associated to Eq. (30) with $\omega_{\mathrm{BR}} = 0$ is derived.

The $(.)_{11}$ term is

$$\bar{Q}_{11} = V_{go}2\sqrt{1 + \frac{\sqrt{2}}{2}}\omega_o = 2.61V_{go}\omega_o, \qquad \omega_o = \sqrt[8]{\frac{W'_{\mathrm{BR}}}{V_{go}}} \tag{C1}$$

The $(.)_{11}$ term is the covariance of the estimation position error (the miss).

The author is not aware of an analytic solution to the associated algebraic Riccati equation of Eq. (30) with $\omega_{\mathrm{BR}} \neq 0$. Therefore, the correction factor approach has been adopted. This approach has the advantage that, although it does not have the advantage of the full analytic solution, it still preserves the main functional dependence of the rms miss distance on the parameters of the problem.

Thus, the covariance of the miss is given by $[\bar{Q}]_{11}$ in Eq. (31).

Substitution of Eqs. (19) and (29) into Eq. (C1) gives Eq. (32).

Appendix D: Derivation of Equation (38)

In [36], the solution of the steady-state Riccati equation associated to Eq. (44) with $\omega_{\mathrm{CBR}} = 0$ is derived. Thus, the solution is as for a third-order system. The $(.)_{11}$ term is

$$\bar{Q}_{11} = V_{go}2\omega_o, \qquad \omega_o = \sqrt[6]{\frac{W_{\mathrm{CBR}}}{V_{go}}} \tag{D1}$$

The $(.)_{11}$ term is the covariance of the estimation position error (the miss).

The author is not aware of an analytic solution of the associated algebraic Riccati equation of Eq. (44) with $\omega_{\mathrm{CBR}} \neq 0$. Therefore, the correction factor approach has been adopted. This approach has the advantage that, although it does not have the advantage of the full analytic solution, it still preserves the main functional dependence of the rms miss distance on the parameters of the problem.

Thus, the covariance of the miss is given by $[\bar{Q}]_{11}$ in Eq. (45).

Substitution of Eqs. (19) and (23) into Eq. (D1) gives Eq. (46).

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